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A Relaxation Method for Solving Nonlinear Stress Equilibrium Problems

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This paper describes a method for modifying explicit finite difference equations for the dynamic motion of a continuum to produce stress relaxation equations for static stress equilibrium. The equations modified are those of Wilkins, but the method is applicable to other finite difference codes. An artificial "stress diffusion" equation is used, in which successive displacements are toward stress equilibrium. Results of a two-dimensional numerical calculation are compared with an analytic solution.

Most methods for static equilibrium problems are restricted to the linear case. Explicit finite difference methods commonly used for dynamic problems can accept quite general material descriptions, such as plastic yielding and tensile cracking. This paper describes a simple modification that converts an explicit dynamic method, such as Wilkins' [1], to a method for static equilibrium allowing the same generality of material description. The modification is especially useful for finding a static stress distribution as an initial condition for a dynamic problem.

In Wilkins' procedure, a two-dimensional continuous medium is divided up into quadrilateral zones. The mesh is Lagrangian, meaning that the intersections of the grid lines, called grid points in this paper, move with the medium, and the quadrilateral zones distort with the medium. Displacements (or spatial coordinates) and velocity are defined at grid points, while stress and strain components are defined in the interiors of zones.

Four distinct steps are performed for each zone in each computational cycle.

1. From stress components in zones surrounding a grid point, find the unbalanced force on the grid point by a finite difference analog of the equation

$$F_i = (\partial \Sigma_{ij}/\partial x_j) + g_i , \tag{1}$$

where F_i is the unbalanced force per unit volume, Σ_{ij} is the stress tensor, and g_i is the body force.

2. Accelerate each grid point in the direction of the unbalanced force by

$$\partial^2 R_i/\partial t^2 = (1/\rho) F_i , \tag{2}$$

where R is the displacement and ρ is density.

3. From new displacements of grid points bounding each zone find increments of strain components for that zone by

$$\dot{\epsilon}_{ij} = \tfrac{1}{2}((\partial \dot{R}_i/\partial x_j) + (\partial \dot{R}_j/\partial x_i)) \tag{3}$$

where $\dot{\epsilon}_{ij}$ is the strain rate tensor.

4. From strain increments find new values of stress components by a material description.

In the modification, the finite difference equations used to represent the relation between displacement of a continuum and strain (Step 3) and the relation between stress gradients and unbalanced force per unit volume (Step 1) are identical to those used by Wilkins for dynamic problems. Only the second of the four steps need be modified to produce a relaxation method for solving equilibrium problems.

In equilibrium, the right side of Eq. (1) is zero:

$$F_i = 0. \tag{4}$$

A way to obtain an iterative scheme to solve for the equilibrium state, Eq. (4), is to introduce a nonphysical "stress diffusion" equation:

$$(\partial R_i/\partial \tau) = (1/\rho) F_i \tag{5}$$

where τ is an artificial variable with the dimensions of time squared, which we shall call "pseudotime" to avoid confusion with the real time t of Eq. (2). It will be shown that the solution of Eq. (5) approaches the solution of the equilibrium equation, Eq. (4), as pseudotime increases. From this point vector notation will be used.

Since the "pseudovelocity," $(\partial \mathbf{R}/\partial \tau)$, is always in the direction of the local force density, the direction of displacement is toward local stress equilibrium. The local "rate" of work, $\mathbf{F} \cdot \partial\mathbf{R}/\partial\tau$, is positive everywhere, so that the total potential energy decreases as pseudotime progresses. Any problem of static stress equilibrium may be formulated as a variational problem in which the solution minimizes total potential energy. The successive finite difference integrations of Eq. (5) correspond conceptually to successive variations in an application of the variational principle.

In an elastic problem the solution minimizing potential energy is unique. In the case of a plastic material, the solution depends on the sequence of states leading up to the equilibrium state. Similarly, the equilibrium state reached in the present relaxation method will depend on the sequence of stress states traversed in the iteration. The accuracy with which a calculated equilibrium state reproduces an actual physical state will depend upon the degree to which the stress history of each zone matches the actual case.

For an elastic material with small displacements the equations of these four distinct steps might be combined to relate displacement of each grid point to displacements of neighboring grid points. The resulting system of linear equations could be solved by any matrix iterative technique. The procedure, as it stands, is equivalent to a point Jacobi relaxation method, for displacement of each point is determined by displacements of neighboring points in the previous iteration. The stability criterion for the pseudotime step discussed later ensures that the relaxation factor does not exceed two.

The cost in computer time for keeping the steps distinct is considerable. The method effectively recalculates the matrix elements in each iteration. Also, without combining the steps it is not possible to use faster iterative techniques such as Gauss–Seidel overrelaxation. However, there is much to be gained by keeping the steps distinct, for arbitrary stress-strain laws can be used.

The procedure may be analyzed further for the special case of an elastic material. Consider the case of small displacements in an isotropic linear elastic solid with no body forces acting. Elasticity theory gives the appropriate form of Eq. (5) in Cartesian coordinates,

$$(\lambda + \mu)\, \nabla(\nabla \cdot \mathbf{R}) + \mu\nabla^2\mathbf{R} = \rho(\partial\mathbf{R}/\partial\tau), \tag{6}$$

where λ and μ are the Lamé elastic constants. Forming the curl and divergence of Eq. (6), and neglecting spatial variations of the mass density, one gets

$$\partial(\nabla \times \mathbf{R})/\partial\tau = C_T{}^2\nabla^2(\nabla \times \mathbf{R}), \qquad C_T{}^2 = \mu/\rho; \tag{7}$$

$$\partial(\nabla \cdot \mathbf{R})/\partial\tau = C_L{}^2\nabla^2(\nabla \cdot \mathbf{R}), \qquad C_L{}^2 = (\lambda + 2\mu)/\rho. \tag{8}$$

These two equations show that the curl and divergence of the displacement vector $\mathbf{R}$, satisfy diffusion equations, with diffusion coefficients equal to the squares of the transverse and longitudinal sound speeds of the material. The essential difference, then, between the solutions of the dynamic equation of motion, Eq. (2), and the static stress relaxation equation, Eq. (5), is that in the dynamic case stress waves propagate through the body and reflect from the boundaries, while in the static case, stress waves diffuse through the body toward an equilibrium state.

In the nonlinear case, Eq. (5) will be a nonlinear diffusion equation. During the step-by-step integration of this equation, the diffusion equation stability criterion must be used. Taking the square of the longitudinal sound speed C_L^2, as the diffusion coefficient, since this dictates a more conservative time step than C_T^2, this criterion is

$$\Delta\tau \leqslant l^2/2C_L^2, \tag{9}$$

where l is a length related to the size of the zones. For a two-dimensional quadrilateral zone, for example, l is taken to be the area of the zone divided by the longest diagonal. The minimum $\Delta\tau$ found by applying Eq. (9) to all zones during one cycle of the computation is used as the time step for the next cycle. The static time step $\Delta\tau$, is related to the time step Δt, appropriate for a dynamic case with the same instantaneous solution, by

$$\Delta\tau = \tfrac{1}{2}(\Delta t)^2. \tag{10}$$

An estimate of the number of cycles of iteration required for convergence to equilibrium can be made. In a diffusion problem with a diffusion coefficient of C_L^2 the characteristic relaxation time T, of a body with a typical dimension L, is of the order

$$T \cong L^2/2C_L^2. \tag{11}$$

With $\Delta\tau$ given by Eq. (9), it will take on the order of

$$N = (L/l)^2 \tag{12}$$

cycles for the longest wavelength component of the error to decay.

Stress boundary conditions are obtained, as in Wilkins, by applying appropriate stresses to the boundary zones. Displacement boundary conditions are obtained by moving the boundary points of the undisturbed object to the desired final displacement as the iteration proceeds. In the case of a plastic material, the boundary points should be moved slowly enough that boundary stresses do not exceed their equilibrium values. Otherwise, excess plastic yielding will occur at the boundary.

Some insight into the stress diffusion equation can be gained by deriving it in another way. Suppose that the dynamic equation of motion, Eq. (2), is used to compute the motion of a continuum, but that at frequent intervals all velocities are set equal to zero, in order to avoid "over-shooting" equilibrium. Then the stresses and displacements in the body ought to approach equilibrium, if the boundary conditions on the problem allow for an equilibrium state. This method of converging to equilibrium was used by Maenchen and Sack [2], who wanted to compare their dynamic TENSOR code's treatment of plasticity with an analytic solution, but only had available a static solution. They set all velocities equal to zero approximately twenty times during their calculation, and were able to get close agreement with the analytic solution.

If this procedure is applied in its limit, where all velocities are set equal to zero at the beginning of each cycle, it is equivalent to solving the stress diffusion equation, Eq. (5). The use of the stress diffusion equation is more amenable to theoretical analysis since it is equivalent to the familiar Jacobi relaxation method.

To illustrate the stress diffusion relaxation method, the results of two numerical calculations are presented. The results of the first calculation, the indentation of an elastic solid by a rigid circular punch, are compared to its analytic solution. The second calculation is the same as the first, except that the effects of plasticity are included.

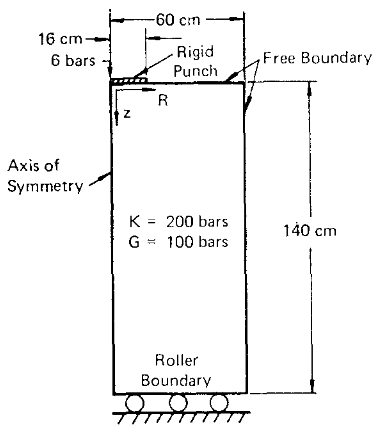

FIG. 1. Geometry and boundary conditions.

This problem is of engineering importance in soil mechanics with respect to the safety of foundations, where the rigid punch represents a foundation footing and the semi-infinite solid is the soil upon which it rests. An analytic solution to the elastic problem is given by Sneddon [3]. The geometry and boundary conditions chosen for the numerical calculation are shown in Fig. 1. The stresses in all zones were zero at the start of the calculation. The grid points beneath the punch were gradually displaced downward during the calculation in such a way that the vertical stress applied by the center of the punch was always six bars. The problem was run for 1000 cycles. During the last 100 cycles the stresses changed by only a few percent.

In Fig. 2 the distribution of vertical stress in the layer of zones beneath the punch is compared with the analytic solution at that depth, and in Fig. 3 the distribution of vertical stress at a depth of approximately one punch radius is compared with the

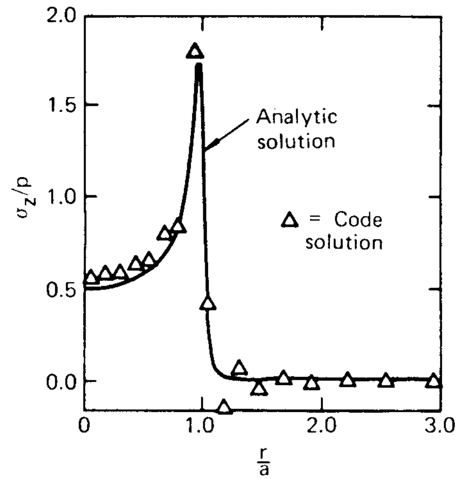

FIG. 2. Radial distribution of vertical stress at a depth of 0.0625 a, where a is the punch radius of 16 cm. The curve is normalized to the average vertical stress p, of 12 bars.

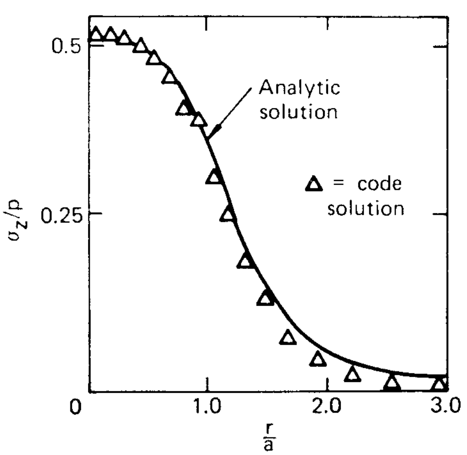

FIG. 3. Radial distribution of stress at a depth of 0.94 a, where a is the punch radius of 16 cm.

analytic solution. The numerical solutions agree with the analytic solution as closely as can be reasonably expected. The stress tensor at the end of the calculation is shown in Fig. 4. We see the vertical stress increasing near the edge of the punch as it does in the analytic solution.

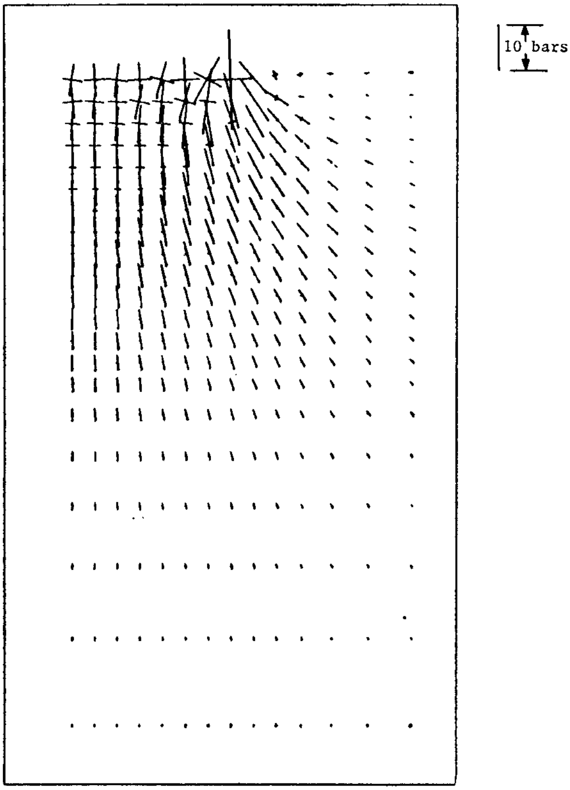

FIG. 4. Equilibrium stress tensor in the elastic case.

The second calculation is the same as the first, except that the solid was allowed to yield plastically. This calculation illustrates the application of the relaxation method to a nonlinear problem. A Mohr–Coulomb yield model was used to describe the plastic yielding, with a coefficient of friction of 1.5. This problem converged somewhat faster than the first calculation. The stress distribution after 1000 cycles of computation is shown in Fig. 5. The vertical stress under the punch is nearly uniform, and at greater depths the stress is not spread out as much as in the elastic case.

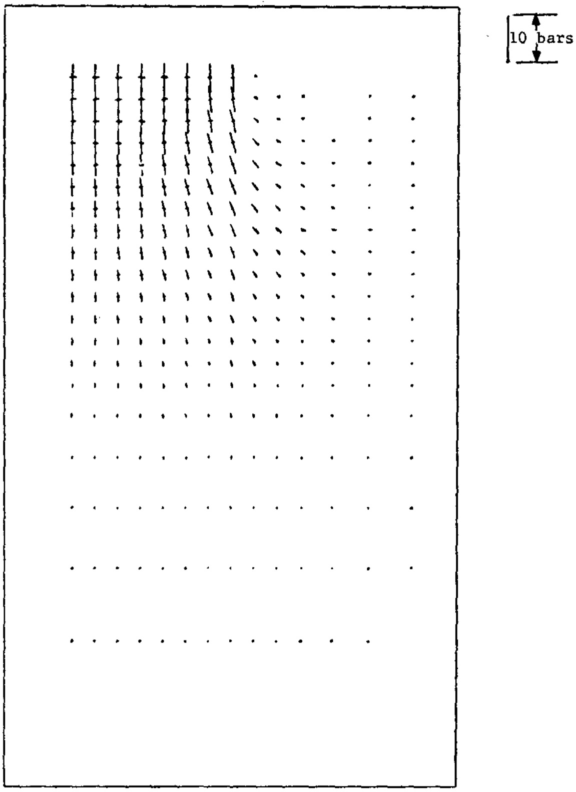

FIG. 5. Equilibrium stress tensor in the plastic case.

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